Study material of B.Sc.(Semester - I)
US01CMTH02
(Higher Order Derivatives)
Prepared by Nilesh Y. Patel
Head,Mathematics Department,V.P.and R.P.T.P.Science College

UNIT-1

1. Higher order derivatives of standard functions

1.1. Definition. Let $f:(a,b)\to\mathbb{R}$ be differentiable on (a,b). If f' is differentiable, then we say that f is twice differentiable and f'' is called the *second derivative* of f. f'' is also denoted by $\frac{d^2f}{dx^2}$. Further, suppose the $(n-1)^{th}$ derivative $f^{(n-1)}$ of f exists. If $f^{(n-1)}$ is differentiable on (a,b), then its derivative is called the n^{th} derivative of f and is denoted by $f^{(n)}$ or $\frac{d^nf}{dx^n}$. If the function f is expressed in terms of $y=f(x)$, then the successive derivatives of y are denoted by $y_1,y_2,\ldots,y_n$ or $\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots,\frac{d^ny}{dx^n}$.

The following proposition describes the formulae for n^{th} order derivatives of some standard functions.

1.2. Proposition. *Let $a,b,c\in\mathbb{R}$.Then prove the following*

(1) *For an integer m if $y=(ax+b)^m$, then*
$$y_n=m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n}.$$
(2) *If $y=(ax+b)^m$, with $m\in\mathbb{N}$, then the above reduces to*
$$y_n=\frac{m!}{(m-n)!}a^n(ax+b)^{m-n}.$$
(3) *If $y=(ax+b)^{-1}$, then*
$$y_n=\frac{(-1)^n n!a^n}{(ax+b)^{n+1}}.$$
(4) *For $y=\log(ax+b)$,*
$$y_n=\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}.$$
(5) *For $y=a^{mx}$,* $\quad y_n=m^n(\log a)^n a^{mx}$.

(6) *For $y=e^{mx}$,* $\quad y_n=m^ne^{mx}$.

(7) *For $y=\cos(ax+b)$,* $\quad y_n=a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$.

(8) *For $y=\sin(ax+b)$,* $\quad y_n=a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$.

(9) *For $y=e^{ax}\cos(bx+c)$,* $\quad y_n=r^ne^{ax}\cos(bx+c+n\varphi)$, *where* $r=\sqrt{a^2+b^2}$, $\varphi=\tan^{-1}\left(\frac{b}{a}\right)$.

(10) *For $y=e^{ax}\sin(bx+c)$,* $\quad y_n=r^ne^{ax}\sin(bx+c+n\varphi)$, *where* $r=\sqrt{a^2+b^2}$, $\varphi=\tan^{-1}\left(\frac{b}{a}\right)$.

PROOF. (1) We prove this by induction. Indeed, $y_1 = ma(ax+b)^{m-1}$.
Assume that $y_n = m(m-1)(m-2)\cdots(m-n+1)a^n(ax+b)^{m-n}$ for a fixed $n \in \mathbb{N}$. Then

$$\begin{aligned}
y_{n+1} = \frac{dy_n}{dx} &= \frac{d}{dx}(m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n}) \\
&= m(m-1)\cdots(m-n+1)a^n\frac{d}{dx}((ax+b)^{m-n}) \\
&= m(m-1)\cdots(m-n+1)a^n(m-n)(ax+b)^{(m-n-1)}a \\
&= m(m-1)\cdots(m-(n+1)+1)a^{n+1}(ax+b)^{(m-(n+1))}.
\end{aligned}$$

(2) and (3) are particular cases of (1).
(4) $y = \log(ax+b)$. So, $y_1 = \frac{1}{ax+b}a = \frac{(-1)^0(0)!\,a^1}{(ax+b)^1} = \frac{(-1)^{1-1}(1-1)!\,a^1}{(ax+b)^1}$.
Now assume that $y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$. Then

$$\begin{aligned}
y_{n+1} = \frac{dy_n}{dx} &= \frac{d}{dx}\left(\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}\right) \\
&= (-1)^{n-1}(n-1)!a^n\frac{d}{dx}\left(\frac{1}{(ax+b)^n}\right) \\
&= (-1)^{n-1}(n-1)!a^n\frac{d}{dx}\,(ax+b)^{-n} \\
&= (-1)^{n-1}(n-1)!a^{n+1}(-n)(ax+b)^{-n-1} \\
&= (-1)^n n!a^{n+1}(ax+b)^{-(n+1)} = \frac{(-1)^n n!a^{n+1}}{(ax+b)^{(n+1)}}\,.
\end{aligned}$$

This proves (4).
(5) $y = a^{mx}$. Then $y_1 = a^{mx}(\log a)m = m(\log a)a^{mx}$. Similarly we get, $y_2 = m^2(\log a)^2a^{mx}$, $y_3 = m^3(\log a)^3a^{mx}$. In general,

$$y_n = m^n(\log a)^n a^{mx}.$$

(6) follows by taking $a = e$ in (5).
(7) $y = \cos(ax+b)$. Then,

$$\begin{aligned}
y_1 &= -a\sin(ax+b) = a\cos\left(ax+b+\frac{\pi}{2}\right) \\
y_2 &= -a^2\sin(ax+b+\frac{\pi}{2}) = a^2\cos\left(ax+b+\frac{2\pi}{2}\right) \\
y_3 &= -a^3\sin(ax+b+\frac{2\pi}{2}) = a^3\cos\left(ax+b+\frac{3\pi}{2}\right) \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
y_n &= -a^n\sin(ax+b+\frac{(n-1)\pi}{2}) = a^n\cos\left(ax+b+\frac{n\pi}{2}\right).
\end{aligned}$$

(8) can be proved similarly.
(9) $y = e^{ax}\cos(bx+c)$.

$$y_1 = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c)$$

$$= e^{ax}(a\cos(bx+c) - b\sin(bx+c)). \tag{1.2.1}$$

Let $a = r\cos\varphi$ and $b = r\sin\varphi$. Then $a^2+b^2 = r^2$ and $\frac{b}{a} = \tan\varphi$. That is, $r = \sqrt{a^2+b^2}$, $\varphi = \tan^{-1}\left(\frac{b}{a}\right)$. Now by (1.2.1),

$$\begin{aligned} y_1 &= e^{ax}(r\cos\varphi\cos(bx+c) - r\sin\varphi\sin(bx+c)) \\ &= re^{ax}\cos(bx+c+\varphi). \end{aligned}$$

Similarly,

$$y_2 = r^2 e^{ax}\cos(bx+c+2\varphi).$$

Continuing in this manner, in general we get,

$$y_n = r^n e^{ax}\cos(bx+c+n\varphi),$$

where $r = \sqrt{a^2+b^2}$, $\varphi = \tan^{-1}\left(\frac{b}{a}\right)$.

(10) can be proved similarly. □

1.3. Example. If $y = \cos mx - \sin mx$, then prove that

$$y_n = m^n\left(1-(-1)^n\sin 2mx\right)^{\frac{1}{2}}.$$

Solution. Here $y = \cos mx - \sin mx$. Hence

$$\begin{aligned} y_n &= m^n\left[\cos\left(mx+\frac{n\pi}{2}\right) - \sin\left(mx+\frac{n\pi}{2}\right)\right] \\ &= m^n\left[\left[\cos\left(mx+\frac{n\pi}{2}\right) - \sin\left(mx+\frac{n\pi}{2}\right)\right]^2\right]^{\frac{1}{2}} \\ &= m^n\left[1 - 2\cos\left(mx+\frac{n\pi}{2}\right)\sin\left(mx+\frac{n\pi}{2}\right)\right]^{\frac{1}{2}} \\ &= m^n\left[1-\sin(2mx+n\pi)\right]^{\frac{1}{2}} \\ &= m^n\left[1-\sin 2mx\cos n\pi - \cos 2mx\sin n\pi\right]^{\frac{1}{2}} \\ &= m^n\left[1-(-1)^n\sin 2mx\right]^{\frac{1}{2}}. \end{aligned}$$

□

1.4. Example. Find y_n for $y = e^{2x}\cos x\sin^2 2x$.

Solution. Here,

$$\begin{aligned} y &= e^{2x}\cos x\sin^2 2x \\ &= e^{2x}\cos x\left(\frac{1-cos4x}{2}\right) \\ &= \frac{1}{2}e^{2x}(\cos x - \cos 4x\cos x) \\ &= \frac{1}{2}e^{2x}(\cos x - \frac{1}{2}(\cos 5x + \cos 3x)) \\ &= \frac{1}{2}e^{2x}\cos x - \frac{1}{4}e^{2x}\cos 5x - \frac{1}{4}e^{2x}\cos 3x \end{aligned}$$

$$\Rightarrow y_n = \frac{5^{n/2}}{2} e^{2x} \cos\left(x + n\tan^{-1}\tfrac{1}{2}\right) - \frac{29^{n/2}}{4} e^{2x} \cos\left(5x + n\tan^{-1}\tfrac{5}{2}\right)$$
$$- \frac{13^{n/2}}{4} e^{2x} \cos\left(3x + n\tan^{-1}\tfrac{3}{2}\right).$$

□

2. Leibniz's rule

2.1. Theorem (Leibniz's rule)**.** *State and prove Leibniz's theorem.*
Statement : Let $\mathbf{u}, \mathbf{v} : E \to \mathbb{R}$ *be sufficiently many times differentiable functions. Then for any* $n \in \mathbb{N}$,

$$(\mathbf{uv})_\mathbf{n} = \mathbf{u_n v} + {}^\mathbf{n}\mathbf{C_1 u_{n-1} v_1} + {}^\mathbf{n}\mathbf{C_2 u_{n-2} v_2} + \cdots + \mathbf{uv_n}.$$

Proof. We prove this theorem by mathematical induction on n. Note that the result for $n = 1$ is $(uv)_1 = u_1 v + uv_1$, which is obviously true. Suppose that the result holds for $n = k$. That is,

$$(uv)_k = u_k v + {}^kC_1 u_{k-1} v_1 + {}^kC_2 u_{k-2} v_2 + \cdots + uv_k.$$

Differentiating this we get,

$$\begin{aligned}(uv)_{k+1} &= u_{k+1} v + u_k v_1 + {}^kC_1 (u_k v_1 + u_{k-1} v_2) + {}^kC_2 (u_{k-1} v_2 + u_{k-2} v_3) \\ &\qquad + \cdots + u_1 v_k + uv_{k+1} \\ &= u_{k+1} v + (1 + {}^kC_1) u_k v_1 + ({}^kC_1 + {}^kC_2) u_{k-1} v_2 \\ &\qquad + ({}^kC_2 + {}^kC_3) u_{k-2} v_3 + \cdots + uv_{k+1} \\ &= u_{k+1} v + {}^{k+1}C_1 u_k v_1 + {}^{k+1}C_2 u_{k-1} v_2 + {}^{k+1}C_3 u_{k-2} v_3 + \cdots + uv_{k+1},\end{aligned}$$

as ${}^kC_{r-1} + {}^kC_r = {}^{k+1}C_r$. Thus the result is true for $n = k + 1$. Hence the result holds for all $n \in \mathbb{N}$. □

Given a product of functions, usually it is a common practice to select the function as u whose n^{th} derivative is known to us.

2.2. Example. If $y = x \log(x - 1)$, then find y_n.

Solution. Let $u = \log(x - 1)$, then by Leibniz's Theorem,

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + uv_n.$$

Hence,

$$\begin{aligned} y_n &= \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} x + n\frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} 1 + 0 + \cdots \\ &= \frac{(-1)^{n-2}(-1)(n-1)(n-2)!\, x}{(x-1)^n} + n\frac{(-1)^{n-2}(n-2)!(x-1)}{(x-1)^n} \\ &= \frac{(-1)^{n-2}(n-2)!}{(x-1)^n}(-x(n-1) + n(x-1)) = \frac{(-1)^{n-2}(n-2)!}{(x-1)^n}(x-n).\end{aligned}$$

□

2.3. Example. If $x = \cos\left(\frac{1}{m}\log y\right)$, then find $y_n(0)$.

Solution. We can write the given relation as

$$y = e^{m\cos^{-1}x}. \tag{2.3.1}$$

Then

$$y_1 = e^{m\cos^{-1}x}\left(-\frac{m}{\sqrt{1-x^2}}\right) \Rightarrow \sqrt{1-x^2}y_1 = -my. \tag{2.3.2}$$

Hence,

$$(1-x^2)y_1^2 = m^2y^2.$$

By differentiating this we get,

$$2(1-x^2)y_1y_2 - 2xy_1^2 = 2m^2yy_1 \Rightarrow (1-x^2)y_2 - xy_1 = m^2y. \tag{2.3.3}$$

Hence by Leibniz's Theorem, we get,

$$\begin{aligned}
&y_{n+2}(1-x^2) + {}^nC_1y_{n+1}(-2x) - 2\ {}^nC_2y_n \\
&\qquad\qquad - y_{n+1}x - {}^nC_1y_n = m^2y_n \\
\Rightarrow&(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n \\
&\qquad\qquad - xy_{n+1} - ny_n - m^2y_n = 0 \\
\Rightarrow&(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0. \qquad (2.3.4)
\end{aligned}$$

From (2.3.1),(2.3.2), (2.3.3) and (2.3.4) we have,

$$\begin{aligned}
y(0) = e^{m\pi/2}; y_1(0) = -me^{m\pi/2}; y_2(0) = m^2e^{m\pi/2};& \\
y_{n+2}(0) = (n^2+m^2)y_n(0).& \qquad (2.3.5)
\end{aligned}$$

Putting $n = 1, 2, 3, \ldots$ in (2.3.5) we get,

$$\begin{aligned}
y_3(0) &= (1^2+m^2)y_1(0) = -e^{m\pi/2}m(m^2+1); \\
y_4(0) &= (2^2+m^2)y_2(0) = e^{m\pi/2}m^2(m^2+2^2); \\
y_5(0) &= (3^2+m^2)y_3(0) = -e^{m\pi/2}m(m^2+1)(m^2+3^2).
\end{aligned}$$

In general,

$$y_n(0) = \begin{cases} e^{m\pi/2}m^2(m^2+2^2)\cdots(m^2+(n-2)^2) & \text{if } n \text{ is even;} \\ -e^{m\pi/2}m(m^2+1^2)\cdots(m^2+(n-2)^2) & n \text{ odd; } n \neq 1. \end{cases}$$

□

2.4. Example. If

$$y = (x - \sqrt{4+x^2})^m, \tag{2.4.1}$$

then find $y_n(0)$.

Solution. By differentiating with respect to x, we get,

$$\begin{aligned}
y_1 &= m(x-\sqrt{4+x^2})^{m-1}\left(1 - \frac{2x}{2\sqrt{4+x^2}}\right) \\
&= m(x-\sqrt{4+x^2})^{m-1}\left(\frac{\sqrt{4+x^2}-x}{\sqrt{4+x^2}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{m(x-\sqrt{4+x^2})^m}{\sqrt{4+x^2}} \\
&= -\frac{my}{\sqrt{4+x^2}} \\
\Rightarrow &\sqrt{4+x^2}y_1 = -my. \qquad (2.4.2)
\end{aligned}
$$

Squaring both the sides,

$$(4+x^2)y_1^2 = m^2y^2,$$

which, on differentiation gives,

$$(4+x^2)2y_1y_2 + 2xy_1^2 = m^2 2yy_1 \Rightarrow (4+x^2)y_2 + xy_1 = m^2y. \qquad (2.4.3)$$

By Leibniz's Theorem, we get,

$$
\begin{aligned}
&y_{n+2}(4+x^2) + {}^nC_1 y_{n+1}(2x) + 2\ {}^nC_2 y_n \\
&\qquad\qquad + y_{n+1}x + {}^nC_1 y_n = m^2 y_n \\
\Rightarrow &(4+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n \\
&\qquad\qquad + xy_{n+1} + ny_n - m^2y_n = 0 \\
\Rightarrow &(4+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0. \qquad (2.4.4)
\end{aligned}
$$

From (2.4.1),(2.4.2), (2.4.3) and (2.4.4) we have,

$$
\begin{aligned}
y(0) &= (-2)^m; \\
y_1(0) &= -m\frac{y(0)}{2} = -\frac{m}{2}(-2)^m; \\
y_2(0) &= \frac{m^2y(0)}{4} = \frac{m^2}{4}(-2)^m; \\
y_{n+2}(0) &= \frac{(m^2-n^2)y_n(0)}{4}. \qquad (2.4.5)
\end{aligned}
$$

Putting $n = 1, 2, 3, \ldots$ in (2.4.5) to get,

$$
\begin{aligned}
y_3(0) &= \frac{(m^2-1^2)y_1(0)}{4} = -\frac{m}{2}(-2)^m\,\frac{(m^2-1^2)}{4}; \\
y_4(0) &= \frac{(m^2-2^2)y_2(0)}{4} = \frac{m^2}{4}(-2)^m\,\frac{(m^2-2^2)}{4}; \\
y_5(0) &= \frac{(m^2-3^2)y_3(0)}{4} = -\frac{m}{2}(-2)^m\,\frac{(m^2-1^2)}{4}\,\frac{(m^2-3^2)}{4}.
\end{aligned}
$$

In general,

$$y_n(0) = \begin{cases} (-2)^m\,\frac{m^2}{4}\,\frac{m^2-2^2}{4}\cdots\frac{m^2-(n-2)^2}{4} & \text{if } n \text{ is even;} \\ -(-2)^m\,\frac{m}{2}\,\frac{(m^2-1^2)}{4}\cdots\frac{(m^2-(n-2)^2)}{4} & n \text{ odd; } n \neq 1. \end{cases}$$

□

2.5. Example. Let $y = (x^2-2)^m$. Find the value of m such that

$$(x^2-2)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$$

Solution. Differentiating y with respect to x, we have,

$$\begin{aligned}
&y_1 = m(x^2-2)^{m-1}2x\\
\Rightarrow&(x^2-2)y_1 = 2mxy\\
\Rightarrow&(x^2-2)y_2 + 2xy_1 = 2m(xy_1+y)\\
\Rightarrow&(x^2-2)y_2 + 2(1-m)xy_1 - 2my = 0.
\end{aligned}$$

Hence, by applying Leibniz's Theorem, we get,

$$\begin{aligned}
&(x^2-2)y_{n+2} + {}^nC_1 2xy_{n+1} + {}^nC_2 2y_n\\
&\qquad\qquad + 2(1-m)(xy_{n+1}+ny_n) - 2my_n = 0\\
\Rightarrow&(x^2-2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n\\
&\qquad\qquad + 2(1-m)(xy_{n+1}+ny_n) - 2my_n = 0\\
\Rightarrow&(x^2-2)y_{n+2} + (n-m+1)2xy_{n+1} + (n^2-2mn+n-2m)y_n = 0\\
\Rightarrow&(x^2-2)y_{n+2} + (n-m+1)2xy_{n+1} + (n-2m)(n+1)y_n = 0.
\end{aligned}$$

Comparing the coefficients of the last equation with the given equation, we find that $m = n$. □

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